KINETIC PROPERTIES OF A BOSE-EINSTEIN GAS AT FINITE TEMPERATURE

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Abstract

We study, in the framework of the Boltzmann-Nordheim equation (BNE), the kinetic properties of a boson gas above the Bose-Einstein transition temperature T_c. The BNE is solved numerically within a new algorithm, that has been tested with exact analytical results for the collision rate of an homogeneous system in thermal equilibrium. In the classical regime ($T > 6\ T_c$), the relaxation time of a quadrupolar deformation in momentum space is proportional to the mean free collision time $\tau_{relax} \sim T^{-1/2}$. Approaching the critical temperature ($T_c < T < 2.7\ T_c$), quantum statistic effects in BNE become dominant, and the collision rate increases dramatically. Nevertheless, this does not affect the relaxation properties of the gas that depend only on the spontaneous collision term in BNE. The relaxation time τ_{relax} is proportional to $(T - T_c)^{-1/2}$, exhibiting a critical slowing down. These phenomena can be experimentally confirmed looking at the damping properties of collective motions induced on trapped atoms. The possibility to observe a transition from collisionless (zero-sound) to hydrodynamic (first-sound) is finally discussed.

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The study of the kinetic properties of a (non-condensate) Bose-Einstein (B-E) gas, being interesting in itself, is becoming particularly important in relation with the recent experimental observations of the Bose-Einstein condensation (BEC) of trapped atoms [1–3] and excitons [4]. In the experimental setups of [1–3], a gas of alkali atoms is trapped and cooled down below the BEC critical temperature ($T \simeq 100\ nK$ for $N \simeq 10^4$ atoms in [1]). The cooling is achieved tuning the trapping magnetic field in order to allow the most energetic particles to escape, with relaxation processes(essentially one- and two-body collisions) driving the system to new equilibrium states at lower temperatures. It is clear that the understanding of the dynamics of the condensation, requires a kinetic model describing the interplay between the time scale associated with the evaporative cooling, the relaxation processes and the emerging of the condensate [5–8]. Such a complete analysis is still lacking [6].

One possible scenario is to assume that the time scales associated with the growth of the condensate are large compared with the relaxation to equilibrium. The condensation will have a quasi-static character allowing the description of the system in the framework of the grand-canonical statistical ensemble [9–11]. On the other hand, for such dilute systems (the mean inter-atomic distance is much greater then the scattering length), the equilibration times can be far longer than the time associated with the growth of the condensate. In this case, non-equilibrium processes would be crucial in understanding the BEC. Yet, the finite life-time of the trapped atoms is in competion with the time necessary for the emergence of the condensate. This particular problem explains the so far unfruitful search of condensation in a trapped gas of Hydrogen atoms [12]. A further intriguing possibility, is the observation of the transition from collisionless (zero-sound) to hydrodynamic (first-sound) modes, that occurs when the number of collisions is enough to reach local thermodynamic equilibrium.

Several theoretical papers address the dynamics of a weakly interacting B-E gas. Levich and Yakhot consider its evolution in the presence of a thermal bath [13], finding that the time for condensation is infinite. Snoke and Wolfe [14] solve numerically the BNE for an homogeneous, thermally isolated, boson gas. They demonstrate that in the classical regime

(high temperatures and/or low densities) the Maxwell-Boltzmann distribution is achieved within few inter-atom scattering times. At sufficiently high densities, in the degenerate regime, the collision rate increases, but the number of collisions necessary to relax the system increases as well. The net result is that the relaxation time remains proportional to the classical mean collision time. Again, the appearance of a B-E condensate (an infinite value of the distribution at zero momentum) is not achieved. However, a different conclusion is reached by Semikoz and Tkachev [15], who found the signature of the B-E condensation in a finite time.

The authors of [14,15] consider an isotropic deformation in momentum space. In this case it is possible to simplify greatly the collision integral in BNE, making feasible the numerical calculations in a grid of discrete energy points.

In this paper we study, as a function of the temperature, both the collision rate of a B-E gas at equilibrium and the relaxation time of a quadrupolar deformation in the momentum space. Differently from the cases studied in [14,15] we need to solve the BNE in the full momentum space, without assuming the isotropy of the system. To this end, we develop a new numerical algorithm, that we test with analytical results. Our analsys indicates that for a quadrupolar (non-isotropic) deformation the relaxation times diverges as $\tau_{relax} \propto (T - T_c)^{-1/2}$, exhibiting a critical slowing down. This could seem in contradiction with the dramatic increase of the collision rate that occurs approaching the critical temperature. However, the effect is easily understood as due to the cancellations between the stimulated "gain" and "loss" collision terms in the BNE, with the relaxation time that depends only on the "classical" spontaneous collision term. As a consequence, close to the critical temperature $T < 2.7\, T_c$ the system exhibits zero-sound modes, and the simple criteria given in the literature that state the hidrodynamic limit occurs when $\omega\tau_{coll} << 1$, with ω the frequency of the collective motions, clearly breaks down.

We note that the BNE approach cannot describe the onset of the B-E phase transition. In fact, the emerging of coherence (with all atoms in the condensate having a common, well defined phase) requires higher order correlations [7] not included in BNE, that is derived in

the random phase (semiclassical) approximation [6].

The experimental study of kinetic properties of the B-E gas is quite active. Collective motions of a trapped condensate have been induced using time-dependent magnetic fields with different multipolarities [16,17]. The frequencies [18–20] and the relaxation times [19] of these resonances have been studied in detail, and the interest is now focusing on their finite-temperature properties [21].

The modification of the classical Boltzmann equation in order to take into account the effect of Bose statistics in the collision integral was first suggested by Nordheim [22]:

$$\frac{\partial n}{\partial t} + \frac{\vec{p}}{m}\frac{\partial n}{\partial \vec{r}} - \frac{\partial U}{\partial \vec{r}}\ \frac{\partial n}{\partial \vec{p}} = (\frac{\partial n}{\partial t})_{col}$$

$$(\frac{\partial n}{\partial t})_{col} = \frac{g}{(2\pi\hbar)^6}\int d\vec{p}_2\ d\vec{p}_3\ d\vec{p}_4\ W\ \delta^3(\vec{p}+\vec{p}_2-\vec{p}_3-\vec{p}_4)\ \delta(\epsilon+\epsilon_2-\epsilon_3-\epsilon_4)$$

$$n(\vec{r},\vec{p})n(\vec{r},\vec{p}_2)[1+n(\vec{r},\vec{p}_3)][1+n(\vec{r},\vec{p}_4)] - n(\vec{r},\vec{p}_3)n(\vec{r},\vec{p}_4)[1+n(\vec{r},\vec{p})][1+n(\vec{r},\vec{p}_2)] \qquad (1)$$

with g the spin degeneracy that, later on, will be set $g = 1$. The collision rate W is related with the scattering cross-section as

$$W = \frac{8\pi^2\sigma\hbar^3}{m^2} \qquad (2)$$

where $\sigma = 8\pi a^2$, with a the scattering length. Here we develop a Boson Monte Carlo Dynamics (BMD) method to solve this equation for the homogeneous case (that can be generalized to solve the BNE eq.(1) in the full phase-space). We parametrize the one-body distribution function as a sum of "weighted-test-particles" (WTP):

$$n(\vec{p},t) = c\sum_{i=1}^{N_{TP}}\ w(\vec{p}-\vec{p}_i(t))$$

$$c = \frac{\rho_0}{N_{TP}}(2\pi\hbar)^3 \qquad (3)$$

where ρ_0 is the density of the system, and N_{TP} is the total number of "test-particles". We choose the weight functions $w(\vec{p}-\vec{p}_i) = \delta(\vec{p}-\vec{p}_i)$ where $p_i(t)$ is the time dependent position in momentum space of the test-particle i.

The time evolution of the particles swarm proceeds along the following steps:

1) the test-particles are randomly distributed in momentum space according to the initial distribution $n(\vec{p}, t = 0)$

2) at each time-step Δt, we randomly choose two test-particles i, j, and we define a collision probability as:

$$P_{coll} = 1 - exp[-\frac{\|\vec{p}_i - \vec{p}_j\|}{m} \rho_0 \ \sigma \ F(\vec{p}_i^{\,\prime}, \vec{p}_j^{\,\prime}) \ \Delta t] \qquad (4)$$

where $F = [1 + n(\vec{p}_i^{\,\prime})][1 + n(\vec{p}_j^{\,\prime})]$.

3) A random number $R = \{0, 1\}$ is compared with P_{coll}: if $R \leq P_{coll}$ the collision is accepted.

4) The new momenta $\vec{p}_i^{\,\prime}$ and $\vec{p}_j^{\,\prime}$ that the two particles would have if the collision were accepted are randomly chosen imposing the conservation of energy and momentum.

4) Repeat the process from 2) until all test particles have had the chance to collide in that time step.

The total energy, momentum and number of particles are exactly conserved at all time-steps. It is clear that $\Delta t \ \ (\sim 0.01 \ s)$ and $N_{tp} \ \ (\sim 1000)$ should be chosen to reach the convergence of the results. It is especially important for the time step Δt to be small enough in order to ensure that the number of two-body collisions (that diverges approaching the critical temperature) remains small compared with the total number of test-particles ($N_{coll} << N_{TP}/2$) at each time step. The factor F in eq.(4) gives the stimulated collisions induced by the statistics. For fermions the minus sign should be used instead of the plus sign; $F = 1$ leads to the classical Boltzmann collision integral. We note that similar Monte Carlo methods have been used for fermion [23,24] and boson [25] systems.

The number of two-body collisions per unit time (collision rate) as a function of the temperature is given by:

$$\frac{dN_{coll}}{d\vec{r} \ dt} = \frac{1}{(2\pi\hbar)^9} \int d\vec{p}_1 \ d\vec{p}_2 \ d\vec{p}_3 \ d\vec{p}_4 \ W \ \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \ \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$
$$n(\vec{p})n(\vec{p}_2)[1 + n(\vec{p}_3)][1 + n(\vec{p}_4)] \qquad (5)$$

Expanding the B-E distribution as: $n = (exp[(H - \mu)/T] - 1)^{-1} = \sum_{j=1}^{\infty} exp(j(\mu - H)/T)$, where $H = \frac{p^2}{2m} + U(\vec{r})$ with U the external (or mean-field) potential, we can carry out

analytically all the integral in eq.(5), getting:

$$\frac{dN_{coll}}{d\vec{r}dt} = \frac{\sigma}{(2\pi\hbar)^6}\frac{8}{\pi m}(2\pi mT)^{7/2}\sum_{i,j}\frac{\sqrt{i+j}}{i^2j^2}exp[\frac{(i+j)(\mu-U(\vec{r}))}{T}]$$

$$+2\sum_{i,j,k}\frac{\sqrt{i+j+k}}{ij(i+k)(j+k)}exp[\frac{(i+j+k)(\mu-U(\vec{r}))}{T}]+$$

$$\sum_{i,j,k,l}\frac{\sqrt{i+j+k+l}}{(i+k)(j+k)(i+l)(j+l)}exp[\frac{(i+j+k+l)(\mu-U(\vec{r}))}{T}] \quad (6)$$

Since we are considering homogeneous systems, we put $U(\vec{r}) = const.$ In fig.(1) we compare the analytical results for the normalized collision rate $\frac{dN_{coll}}{dt}$ eq.(6) (dotted line) with the results obtained numerically with BMD (solid line). At temperatures $T > 6\ T_c$, the Bose-Einstein collision rate becomes equal to the classical Maxwell-Boltzmann one (dashed line). At $T = 2.7\ T_c$ the B-E collision rate shows a minimum, and for $T < 2.7\ T_c$ it increases rapidly deviating significantly from the classical Maxwell-Boltzmann trend. The fast increase is due to the stimulated collisions induced by the statistical factors $(1+n)$ in eq.(5). At high temperatures (classical regime), $n << 1$ and these factors are negligible, while, approaching the critical temperature, $n >> 1$ and quantum statistic effects become dominant. To better illustrate this point, we show in fig.(2) the different contributions to the collision rate obtained from eq.(5). The long-dashed line corresponds to the term coming from $n(\vec{p})n(\vec{p}_2)$. This contribution decreases with temperature as it does the classical one (Maxwell-Boltzmann) in fig.(1). The dotted line corresponds to the term $n(\vec{p})n(\vec{p}_2)[n(\vec{p}_3)+n(\vec{p}_4)]$, that shows the deviation from the classical trend. The biggest contribution to the collision rate comes from the term $n(\vec{p})n(\vec{p}_2)n(\vec{p}_3)n(\vec{p}_4)$ (dashed line).

The chemical potential μ is constrained by the normalization condition:

$$\sum_{j=1}^{\infty} exp(j\mu/T)j^{(-3/2)} = (\frac{2\pi\hbar^2}{m})^{\frac{3}{2}}\rho_0 T^{\frac{3}{2}} = 2.6(\frac{T}{T_c})^{\frac{3}{2}} \quad (7)$$

where the critical temperature, obtained setting $\mu = 0$, is $T_c = (2\pi\hbar^2\rho_0/2.6\ m)^{\frac{3}{2}}$.

Eq.(7) shows that $\frac{\mu}{T} = G[\frac{T}{T_c}]$ where G is a universal function independent of the density of the system. This fact allows to calculate from fig.(1) the collision rate for an homogeneous

system with arbitrary density. In particular, for a given $\frac{T}{T_c}$, we have:

$$\frac{dN_{coll}}{d\vec{r}dt} \propto \sigma {T_c}^{7/2} \tag{9}$$

From the experimental point of view it is of particular importance to know the relaxation time τ_{relax}, that is, the characteristic time for an anisotropic distribution to reach the equilibrium. On general grounds, it could be expected that τ_{relax} were proportional to the mean free collision time τ_{coll}.

We define the relaxation time for a quadrupolar deformation as [26]:

$$\frac{1}{\tau_{relax}} = \frac{\int (\frac{\partial n}{\partial t})_{col}\ P_2(\theta)\ d\vec{p}}{\int n(\vec{p})\ P_2(\theta)\ d\vec{p}} \tag{10}$$

To evaluate eq.(10) numerically, we deform the initial equilibrium distribution as:

$$\vec{p} \rightarrow p\ (1 + \alpha P_2(\theta)) \tag{11}$$

where $P_2(\theta) = \frac{1}{2}[3cos^2(\theta) - 1]$ is the Legendre function. We note that for an arbitrary monopolar deformation (that implies only a scaling change in the modulus of the momenta $p \rightarrow p(1+\alpha)$) the Boltzmann-Nordheim collision integral in eq.(1) becomes exactly zero (this holds also for fermions and classical particles). In this case the dynamics is entirely governed by the external (mean-field) potential.

The quadrupolar relaxation time as a function of temperature is shown in Fig.(3) (solid line) for $\alpha = 0.5$. We plot in the same figure the mean free collision time $\tau_{coll} = \frac{dN_{coll}}{dt})^{-1}$, corresponding to the average time between two collisions for the system in the thermal equilibrium. These two characteristic times are proportional for $T/T_c > 3$, but their behavior differs significantly at lower temperatures. For $T/T_c < 2.7$ the collision time drops due to the increase in the collision rate (see fig.1) but, surprisingly enough, the relaxation time increases as:

$$\tau_{rel} \propto (T - T_c)^{-1/2} \tag{2}$$

Note that for a classical Maxwell-Boltzmann distribution the relaxation time goes as $\tau_{relax}^{MB} \propto T^{-1/2}$. The reason for this behavior can be understood looking at the expression of the

collision integral, eq.(1). The two terms involving the product of four distribution functions, $F_4 = n(\vec{r},\vec{p})n(\vec{r},\vec{p}_2)n(\vec{r},\vec{p}_3)n(\vec{r},\vec{p}_4)$ cancel each other exactly for any distribution $n(\vec{r},\vec{p},t)$. On the other hand, it is this term that, in eq.(5), gives the largest contribution to the collision rate at low temperatures. Moreover, the terms involving the product of three distributions, $F_3 = n(\vec{r},\vec{p})n(\vec{r},\vec{p}_2)[n(\vec{r},\vec{p}_3)+n(\vec{r},\vec{p}_4)]$, almost cancel each other for the deformation that we have considered. In fig.(4,a) we show separatedly the contributions of each term of eq.(1) to the relaxation of the initial quadrupolar deformation. The dot-dashed line indicates the time evolution of the quadrupolar momentum considering only the term F_4 in the Monte Carlo simulation: as expected there is no contribution to the relaxation of the system. However, in fig.(4,b), we show how this term gives the largest contribution to the collision rate. The term F_3 gives also a small contribution in fig.(4,a) (dotted line), while it still gives a large contribution to the collision rate in fig.(4,b). Hence, we end up with the conclusion that the stimulated collisions do not contribute to the relaxation of the system. Only the spontaneous collision term ($F = 1$ in eq.(4)) affects the relaxation time, although this gives the smallest contribution to the collision rate (long-dashed line in fig.s(4,a,b)).

As a consequence of these results, we remark that the simple criteria given in the literature that state the transition between zero and first sound occurs when $\omega\ \tau_{coll} << 1$, with ω the frequency of a collective mode, do not apply for a boson gas at $T_c < T < 2.7\ T_c$, where $\tau_{coll} \to 0$, but $\tau_{rel} \to \infty$. Close to the critical temperature, the system should exhibit collisionless modes, with a transition to first sound at higher temperatures, when $\omega\tau_{relax} << 1$.

The predicted trend of τ_{relax} vs. temperature in fig.(3) can be tested experimentally. In particular, the damping of collective quadrupole motions, induced in the trapped gas, should rapidly decrease approaching T_c according eq.(12). In effect the B-E distribution observed experimentally is non-homogeneous, being the boson gas trapped in an harmonic field. However we do expect that the conclusion of this work and the trend predicted by eq.(12) can still give a good approximation provided that the size of the system is much larger than the characteristic scale of inhomogeneity.

We conclude remarking that the rapid increase of τ_{relax} for $T \to T_c$ questions the as-

sumption that the condensation observed experimentally takes place in equilibrated systems. Understanding how the BEC occurs in systems far from equilibrium is one of the most interesting and challenging problems raised by the physics of trapped atoms.

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REFERENCES

[1] M.H.Anderson, J.R.Ensher, M.R.Matthews, C.E.Wieman and E.A.Cornell, Science 269 (1995) 198

[2] K.B.Davis et al.,Phys.Rev.Lett.75, (1995) 3969

[3] C.C.Bradley et al., Phys.Rev.Lett.75 (1995) 1687; 77 (1997) 985

[4] G.M.Kavoulakis, G.Baym and J.P.Wolfe, Phys.Rev.B53 (1996)7227 and ref.s therein

[5] For a review see "Bose-Einstein condensation" edited by A.Griffin, D.W.Snoke, and S.Stringari (Cambridge University Press, Cambridge, England, 1995).

[6] Yu. Kagan contribution in [5].

[7] H.T.C. Stoof contribution in [5].

[8] Gardiner and Zoller, Phys.Rev. A55 (1997) 2902

[9] G.Baym and C.J.Pethick, Phys.Rev.Lett.76 (1996) 6

[10] V.V.Goldman, I.Silvera and A.Legget, Phys. Rev. B24 (1981) 2870

[11] D.A.Huse and ED. Siggia, Jour. of Low Temp. Phys. 46 (1982) 137

[12] C. E. Wieman, Am. J. Phys. 64 (1996) 847

[13] E. Levich and V. Yakho, J. Phys. A11 (1978) 2237

[14] D.W.Snoke and J.P.Wolfe, Phys.Rev.B39 (1989) 4030

[15] D.V.Semikoz and I.I.Tkachev, Phys.Rev.Lett.74 (1995) 3093

[16] D.S. Jin et al., Phys.Rev.Lett.77 (1996) 420

[17] M.O.Meves et al. Phys.Rev.Lett. 77 (1996) 988

[18] Yu.Kagan. et al. Phys.Rev.A54 (1996) 1753R

[19] A. Smerzi and S. Fantoni, Phys. Rev. Lett. 78 (1997) 3589

[20] S. Stringari, Phys. Rev. Lett. 77 (1996) 2360; M. Edwards et al. Phys. Rev. Lett. 77 (1996) 1671

[21] D.S.Jin et al., Phys. Rev. Lett. 78 (1997) 764

[22] C.W.Nordheim, Proc.R.Soc. London, A119 (1928) 689

[23] G.F.Bertsch and S.Gupta, Phys.Rep.160 (1988) 189

[24] A.Bonasera, F.Gulminelli and J.Molitoris, Phys.Rep.243, (1994) 1

[25] G.M.Welke and G.F.Bertsch, Phys.Rev.C45 (1992) 1403; G. F. Bertsch in "Statistical description of transport in plasma, astro- and nuclear physics", edited by J.Misguich, G. Pelletier and P. Schuck, Les Houches, February 1993.

[26] G.Bertsch, Z.Physik A (1978) 103

[27] A.Griffin, Wen-Chin Wu and S. Stringari Phys. Rev. Lett. 78 (1997) 1838 and references therein

FIGURES

FIG. 1. Normalized collision rate as a function of the temperature, calculated numerically (solid line) and analitycally (dotted line). The dashed line shows the collision rate for a classical Maxwell-Boltzmann distribution. The particle density is $\rho = 10^{13} cm^{-3}$ and the collision cross section is $\sigma = 4.4\ 10^{-13} cm^2$.

FIG. 2. Contributions to the collision rate from each term of the collision integral in BNE eq.(1), as explained in the text.

FIG. 3. Comparison between the relaxation time (solid line) and the mean free collision time (dashed line) as a function of temperature.

FIG. 4. (a) Quadrupole moment as a function of time for $T = 100\ nK$. The different lines correspond to contributions coming from each term in the collision integral of BEN eq.(1), as explained in the text (see fig.2). (b) Corresponding collision rates.

COLLISION RATE

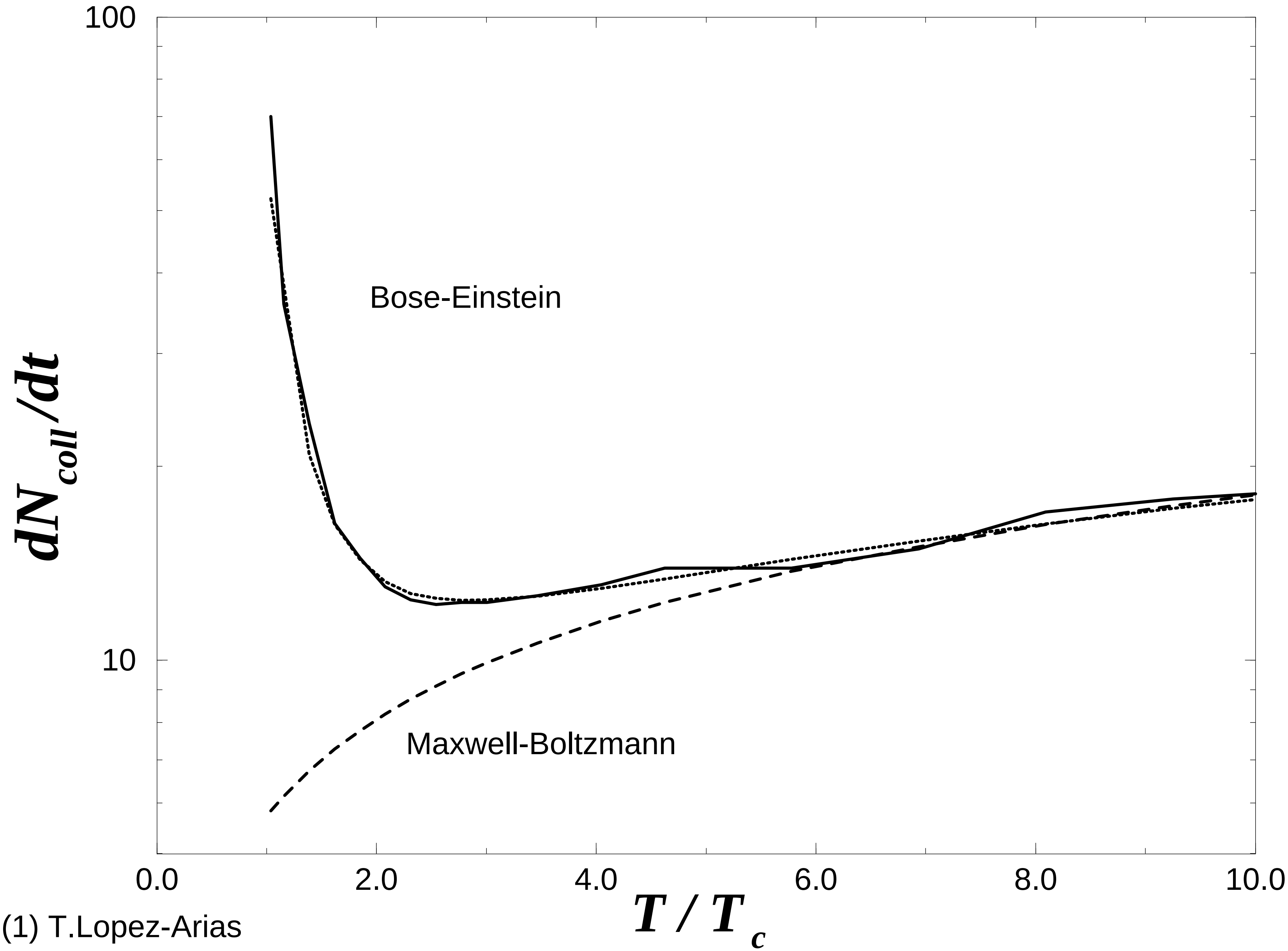

Fig.(1) T.Lopez-Arias

COLLISION RATE

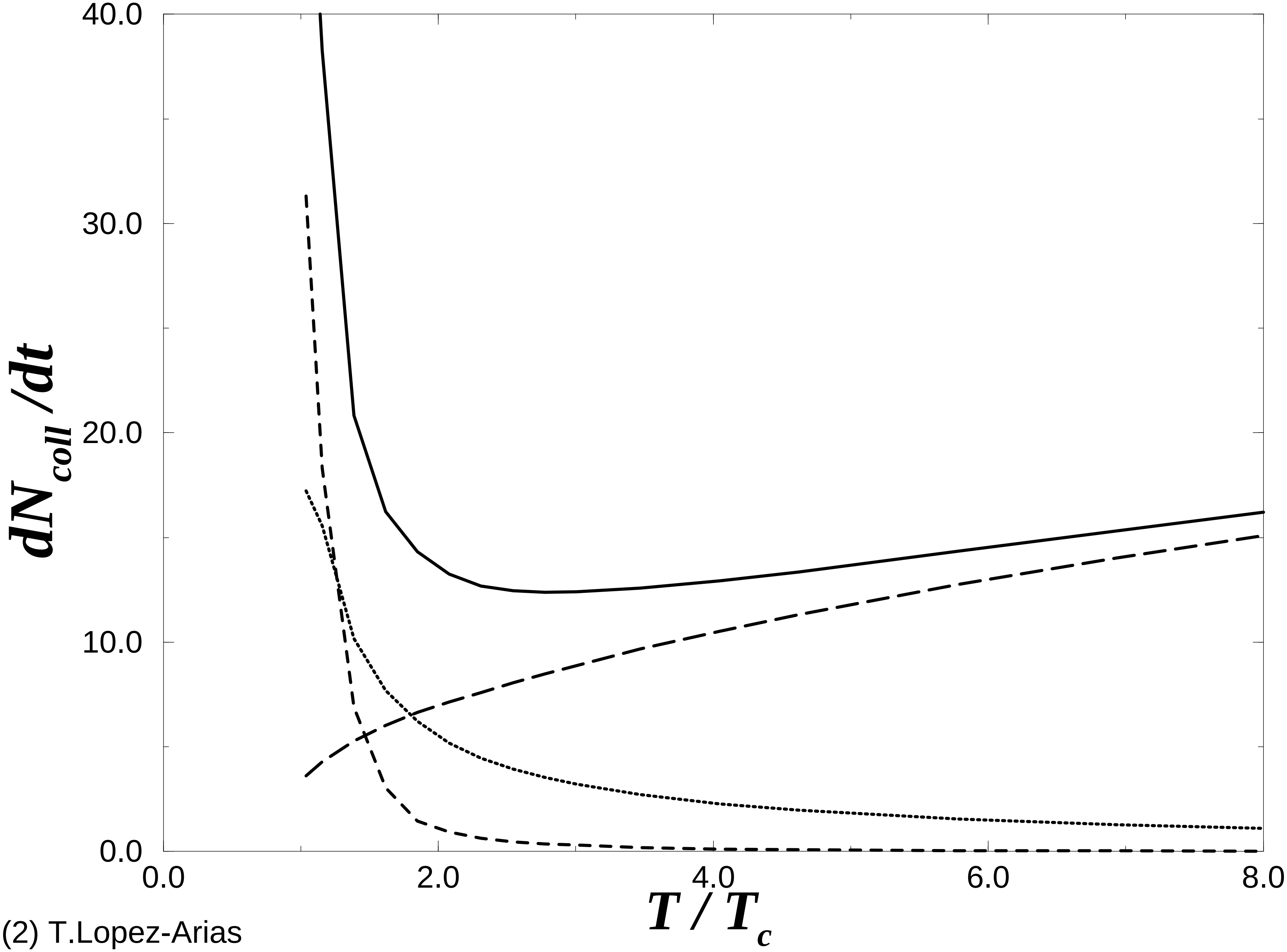

Fig.(2) T.Lopez-Arias

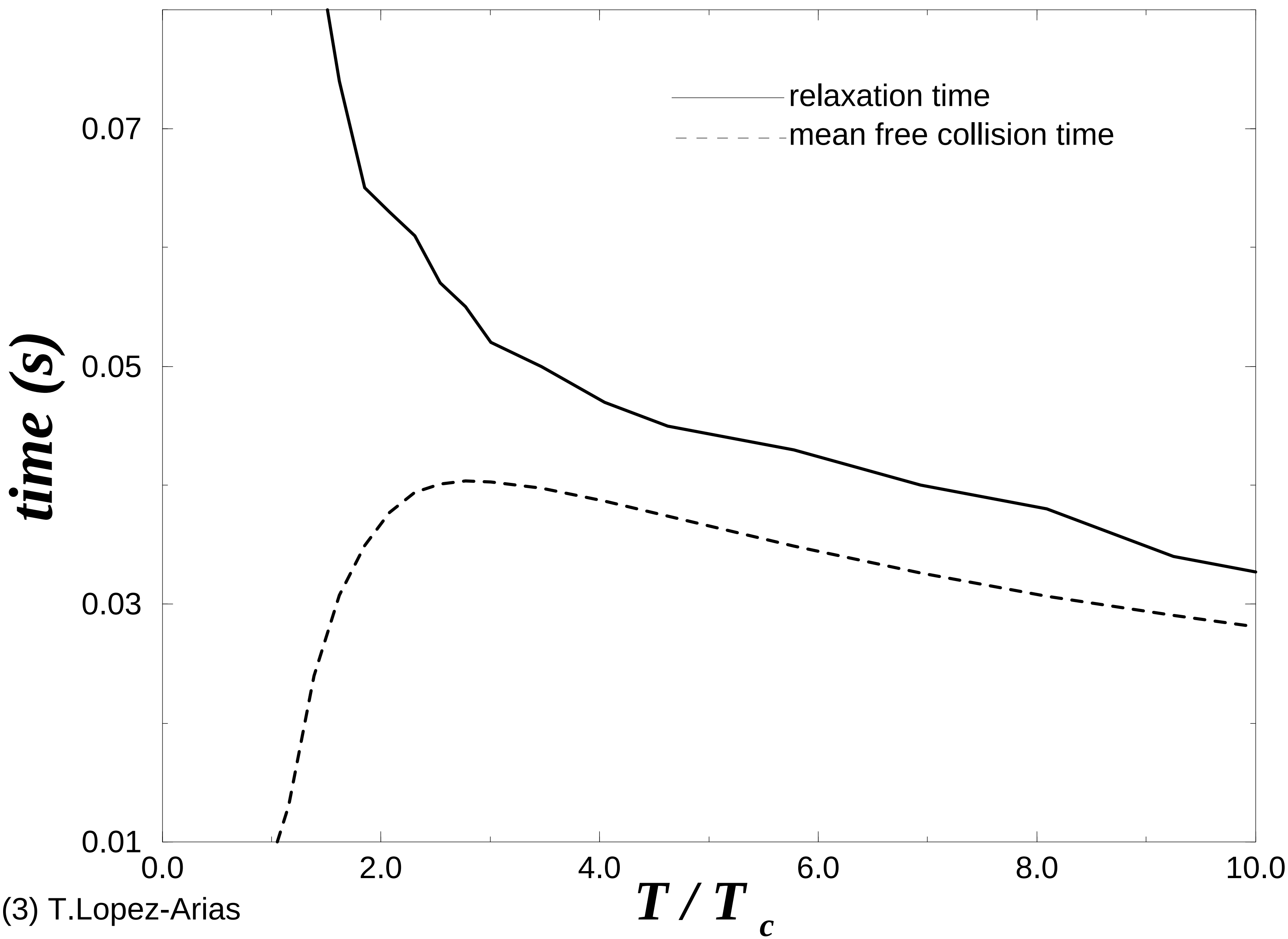

Fig.(3) T.Lopez-Arias

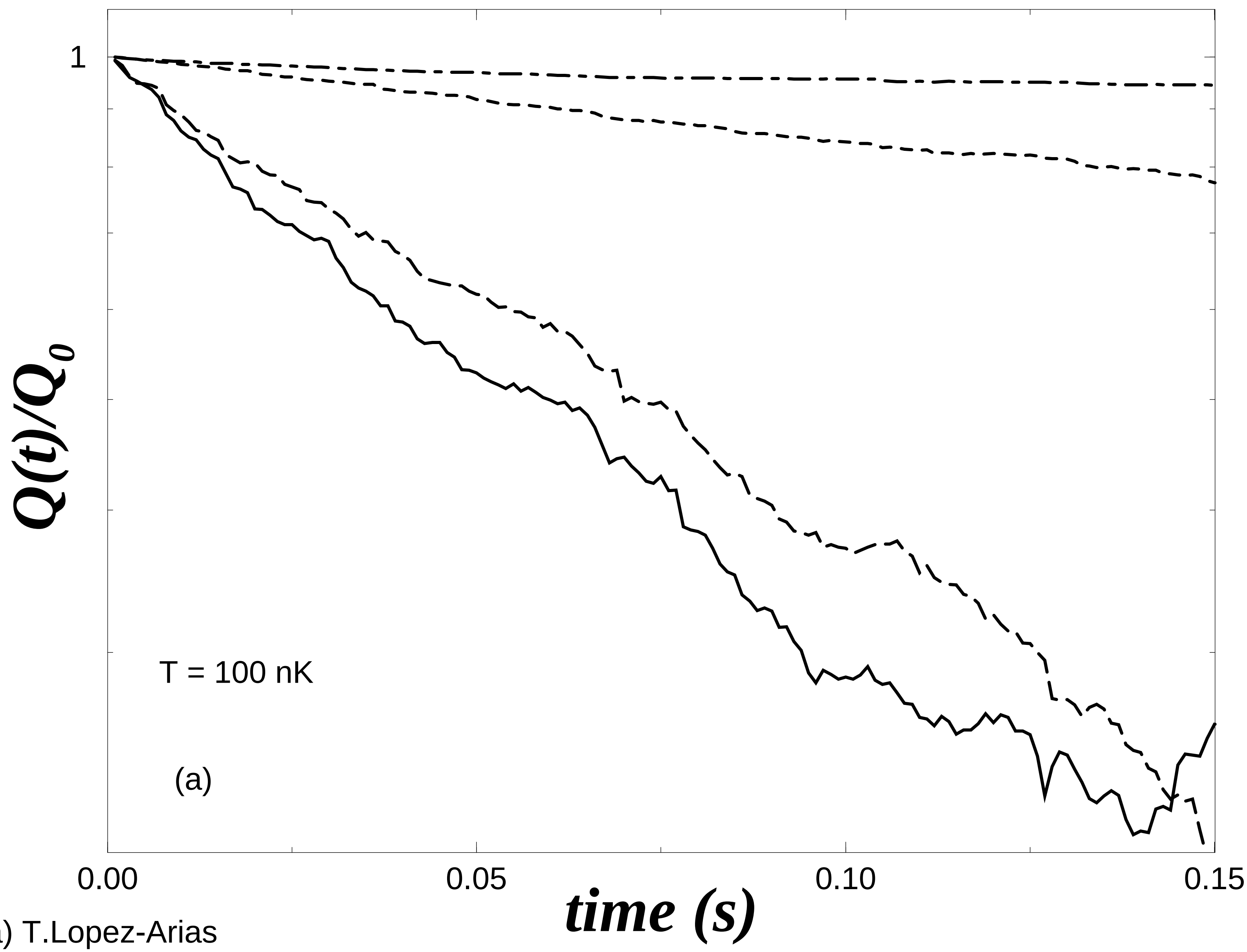

Fig.(4a) T.Lopez-Arias

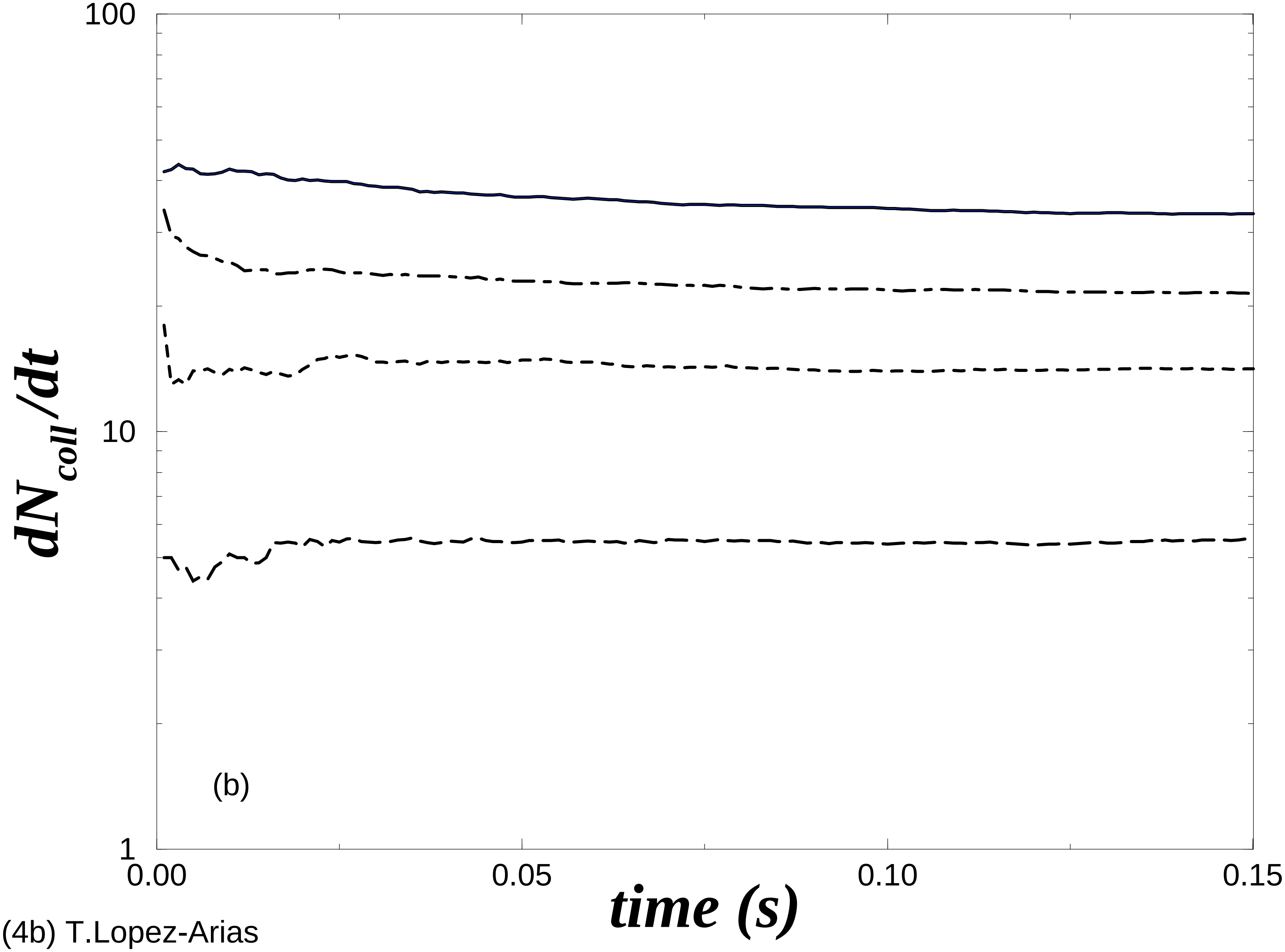

Fig.(4b) T.Lopez-Arias